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Removal of any Two Terms from a Binary Quantic by Linear Transformations.

By Bessie Growe Morrison.

In his "Bericht," p. 179, Meyer has enumerated the various steps in the history of this subject.

More recently, H. B. Newson (Annals of Math., vol. XI, Nos. 3 and 4, 1897) showed that every non-singular quantic of odd degree may be linearly transformed so that its middle term shall vanish.

We propose to discuss those linear transformations by which any two terms of a binary quantic disappear.

§1.—Covariants $C_{n-r,\kappa}$ and $C_{r,n-\kappa}$.

$$\left.\begin{array}{l} r = (1,\ 2 \ldots\ldots n-1) \\ \varkappa = (1,\ 2 \ldots\ldots n-1) \end{array}\right) n - r > \varkappa.$$

If to the quantic $f = \alpha_z^n$ we apply the transformation

$$T\colon \left.\begin{array}{l} z_1 = \lambda_1 z_1' + \mu_1 z_2' \\ z_2 = \lambda_2 z_1' + \mu_2 z_2' \end{array}\right\}, \ \Delta \neq 0,$$

then f takes the form

$$(A)\ f'(z') = \alpha_\lambda^n z_1'^n + \tbinom{n}{1} \alpha_\lambda^{n-1} \alpha_\mu z_1'^{n-1} z_2' + \ldots\ldots \alpha_\mu^n z_2'^n.$$

If by a transformation T any two terms of f' vanish simultaneously, the necessary and sufficient condition is

$$\begin{array}{ll} (1) & \alpha_\lambda^r \alpha_\mu^{n-r} = 0 \\ (2) & \alpha_\lambda^{n-\kappa} \alpha_\mu^\kappa = 0 \end{array} \left\{ \begin{array}{l} r = (0, 1, 2, \ldots\ldots n-1) \\ \varkappa = (0, 1, 2, \ldots\ldots n-1) \end{array}\right) n - r > \varkappa.$$

The condition that (1) and (2) vanish simultaneously is that the resultant in μ, $R_{n-r,\kappa}$, or the resultant in λ, $R_{r,n-\kappa}$, vanish.

Hence the problem is to eliminate λ or μ between (1) and (2).

Either resultant contains the factor f. Let μ be a root of $f = \alpha_z^n$, then $\alpha_\mu^n = 0$.

$$(1) \quad f_{(\mu)^{n-r}} = \alpha_z^r \alpha_\mu^{n-r},$$
$$(2) \quad f_{(\mu)^\kappa} = \alpha_z^{n-\kappa} \alpha_\mu^\kappa.$$

It is evident that (1) and (2) vanish simultaneously for $z = \mu$;

since $$\alpha_\mu^r \alpha_\mu^{n-r} = \alpha_\mu^n = 0,$$
and $$\alpha_\mu^{n-\kappa} \alpha_\mu^\kappa = \alpha_\mu^n = 0.$$

Therefore f is a factor of $R_{n-r,\kappa}$ and similarly if λ is a root of f, then also is f a factor of $R_{r,n-\kappa}$.

Now if λ and μ have values which satisfy $f = \alpha_z^n$, $\alpha_\lambda^n = 0$ and $\alpha_\mu^n = 0$. Hence in A the first and last terms vanish. Since this case is so simple we exclude it by dividing f out of $R_{n-r,\kappa}$ and $R_{r,n-\kappa}$ calling the remaining functions $C_{n-r,\kappa}$ and $C_{r,n-\kappa}$ respectively, where

$$\left\{ \begin{array}{l} r = (1, 2, \ldots . \; n-1) \\ \varkappa = (1, 2, \ldots . \; n-1) \end{array} \right., \; n - r > \varkappa,$$

$R_{n-r,\kappa}$ and $R_{r,n-\kappa}$ are both of degree $[(n-r)(n-\varkappa) + \varkappa r]$,

and $C_{n-r,\kappa}$ and $C_{r,n-\kappa}$ are both of degree $[(n-r)(n-\varkappa) + \varkappa r - n]$

§2.—PROPERTIES OF λ-POINTS AND μ-POINTS.

From the consideration of

$$\left. \begin{array}{ll} (1) & f_{(\mu)^{n-r}} = \alpha_z^r \alpha_\mu^{n-r} \\ (2) & f_{(\mu)^\kappa} \quad = \alpha_z^{n-\kappa} \alpha_\mu^\kappa \end{array} \right\}$$

and

$$\left. \begin{array}{ll} (3) & f_{(\lambda)^r} \quad = \alpha_z^{n-r} \alpha_\lambda^r \\ (4) & f_{(\lambda)^{n-\kappa}} = \alpha_z^\kappa \alpha_\lambda^{n-\kappa} \end{array} \right\}$$

it can be shown that the z-points, or common values of (1) and (2), are the λ-points $R_{r,n-\kappa}$, and the z-points, or common values of (3) and (4), are the μ-points of $R_{n-r,\kappa}$; i. e. if the μ's are the roots of $R_{n-r,\kappa}$ and the λ's the roots of $R_{r,n-\kappa}$, then

$$(5) \quad \alpha_\lambda^r \alpha_\mu^{n-r} = 0,$$
$$(6) \quad \alpha_\lambda^{n-\kappa} \alpha_\mu^\kappa = 0.$$

Similarly if the λ's are the roots of $R_{n-r,\kappa}$ and the μ's the roots of $R_{r,n-\kappa}$ then

$$(7) \quad \alpha_\lambda^{n-r}\alpha_\mu^r = 0.$$
$$(8) \quad \alpha_\lambda^\kappa \alpha_\mu^{n-\kappa} = 0.$$

There is a one-to-one correspondence between the μ's and λ's: For the root μ_1 of the resultant $R_{n-r,\kappa}$ there is a common root λ_1 to (1) and (2); also for the root λ_1 of $R_{r,n-\kappa}$ (3) and (4), have the root μ_1 in common.

Give to μ in T the values of the roots of $C_{n-r,\kappa}$ and to λ the values of corresponding roots of $C_{r,n-\kappa}$, calling the resulting transformations $T_{n-r,\kappa}$, then by $T_{n-r,\kappa}$, the $(n-r+1)^{st}$ and $(\kappa+1)^{st}$ terms of f vanish.

Interchange λ's and μ's, obtaining $T_{r,n-\kappa}$, and by these transformations the $(r+1)^{st}$ and $(n-\kappa+1)^{st}$ terms of f' vanish.

$T_{n-r,\kappa}$ and $T_{r,n-\kappa}$ each consist of $[(n-r)(n-\kappa)+\kappa r-n]$ transformations.

Case $\kappa = r$; $R_{n-r,\kappa}$ and $R_{r,n-\kappa}$ coincide and we call this function $\overline{C_r}$, which is of degree $2r(r-n)+n(n-1)$, an even number.

It is easy to show that the roots of $\overline{C_r}$ go in pairs which have reciprocal relations; the root common to r^{th} and $(n-r)^{th}$ polars of μ_1 is λ_1 and the root common to $(n-r)^{th}$ and r^{th} polars of λ_1 is μ_1.

This reciprocal relation we call correspondence, and denote the corresponding roots by:

$$\mu_p, \lambda_p \left(p = 1, 2, \ldots . \frac{2r(r-n)+n(n-1)}{2}\right).$$

Give to λ and μ in T values of pairs of corresponding roots of $\overline{C_r}$ and we get a set of $2r(r-n)+n(n-1)$ transformations $\overline{T_r}$, by which the $(r+1)^{st}$ and $(n-r+1)^{st}$ terms of f' vanish.

§3.—Form of $C_{n-r,\kappa}$ and $C_{r,n-\kappa}$.

1) $r = 1$

$$(1) \quad f_{(\mu)^{n-1}} = \alpha_z \alpha_\mu^{n-1},$$
$$(2) \quad f_{(\mu)^\kappa} = \alpha_z^{n-\kappa}\alpha_\mu^\kappa.$$

To compute $R_{n-r,\kappa}$ in symbolic notation put

$$z_1 = \rho\beta_\mu^{n-1}\beta_2,$$
$$z_2 = -\rho\beta_\mu^{n-1}\beta_1,$$

then $$R_{n-1,\kappa} = \alpha_\mu^\kappa \left\{ \begin{array}{l} [\beta_\mu^{n-1}\gamma_\mu^{n-1}\delta_\mu^{n-1} \ldots\ldots \text{ to } (n-\varkappa) \text{ factors}]. \\ [(\alpha\beta)(\alpha\gamma)(\alpha\delta) \ldots\ldots \text{ to } (n-\varkappa) \text{ factors}]. \end{array} \right\}.$$

2) $r = \varkappa = 1$.

$$\overline{R_1} = \alpha_\mu [\beta_\mu^{n-1}\gamma_\mu^{n-1} \ldots\ldots \text{ to } (n-1) \text{ factors}]\,[(\alpha\beta)(\alpha\gamma) \ldots\ldots \text{ to } (n-1) \text{ factors}].$$

3) $r = 2$, $(n-\varkappa)$ even*

$$R_{n-2,\kappa} = \alpha_\mu^\kappa \alpha'^\kappa_\mu (\alpha\alpha')^{n-\kappa}\,[\beta_\mu^{n-2}\beta'^{n-2}_\mu(\beta\beta')^2\gamma_\mu^{n-2}\gamma'^{n-2}_\mu(\gamma\gamma')^2 \ldots\ldots \text{ to } \frac{n-\varkappa}{2} \text{ factors}].$$

$$-\frac{n-\varkappa}{2}\,.\,\alpha_\mu^\kappa\alpha'^\kappa_\mu(\alpha\alpha')^{n-\kappa-2}.\,\beta_\mu^{n-2}\beta'^{n-2}_\mu(\alpha\beta)^2(\alpha'\beta')^2\Big[\gamma_\mu^{n-2}\gamma'^{n-2}_\mu$$
$$\times(\gamma\gamma')^2 \ldots\ldots \text{ to } \frac{n-\varkappa-2}{2} \text{ factors}\Big].$$

$$+\frac{(n-\varkappa)(n-\varkappa-3)}{2^2.\,2!}\,\alpha_\mu^\kappa\alpha'^\kappa_\mu(\alpha\alpha')^{n-\kappa-4}\,\beta_\mu^{n-2}\beta'^{n-2}_\mu\gamma_\mu^{n-2}\gamma'^{n-2}_\mu(\alpha\beta)(\alpha'\gamma)$$
$$(\alpha\beta')(\alpha'\gamma')\,.\,\Big[\delta_\mu^{n-2}\delta'^{n-2}_\mu(\delta\delta')^2 \ldots\ldots \text{ to } \frac{n-\varkappa-4}{2} \text{ factors}\Big]$$

$+ \ldots\ldots$ etc. $+ \ldots\ldots$

$$+\ \frac{(-1)^{\frac{n-\kappa}{2}}}{(2)^{\frac{n-\kappa-2}{2}}} \left\{ \alpha_\mu^\kappa\alpha'^\kappa_\mu \left[\begin{array}{l} .\,\beta_\mu^{n-2}\beta'^{n-2}_\mu(\alpha\beta)^2(\alpha\beta')^2 \ldots\ldots \text{ to } \frac{n-\varkappa}{2} \text{ factors} \\ .\,\gamma_\mu^{\mu-2}\gamma'^{n-2}_\mu(\alpha'\gamma)^2(\alpha'\gamma')^2 \ldots\ldots \text{ to } \frac{n-\varkappa}{2} \text{ factors} \end{array} \right] \right\}.$$

4) Means of obtaining $C_{n-r,\kappa}$ or $C_{r,n-\kappa}$ when formulæ become complicated.

a) Let a be a root of $C_{n-r,\kappa}$ and c the corresponding root of $C_{r,n-\kappa}$. In T give μ and λ these values respectively,

then $$z_1 = c_1 z_1' + a_1 z_2',$$
$$z_2 = c_2 z_1' + a_2 z_2',$$

which is a transformation of the set $T_{n-r,\kappa}$.

Apply this transformation to,

$$C_{r,n-\kappa} = (cz)(\ldots\ldots\ldots).$$

then $$C'_{r,n-\kappa} = \rho_1 z_2'(\ldots\ldots\ldots).$$

Therefore the first coefficient of $C'_{r,n-\kappa}$ must be zero.

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* See "Resultant of Binary Quadric and n-ic" by Professor H. S. White, Bulletin of A. M. Soc., 2nd series, Vol. I, No. I. Also Clebsch, "Binare Formen," pp. 84–88. In this paper Professor White points out the changes necessary if we assume the order of f as odd, and we find that his formula would become much more complex, hence our formula would be extremely so.

b) Let a and c be two corresponding roots of $\overline{C}_r$, then

$$z_1 = c_1 z_1' + a_1 z_2'$$
$$z_2 = c_2 z_1' + a_2 z_2'.$$

belongs to $\overline{T}_r$.

Apply this transformation to

$$\overline{C}_r = (az)(cz)(\cdots\cdots\cdots),$$

then
$$\overline{C}_r' = \rho_2 z_1' z_2' \ (\cdots\cdots\cdots).$$

Therefore the first and last coefficients of $\overline{C}_r$ must be zero. To sum up for future reference:

by $T_{n-r,\varkappa}$ the $(n - r + 1)^{st}$ and $(\varkappa + 1)^{st}$ terms of f' vanish,
by $T_{n-r,\varkappa}$ the last term of $C_{n-r,\varkappa}$ and the first term of $C_{r,n-\varkappa}$ vanish,
by $T_{r,n-\varkappa}$ the $(r + 1)^{st}$ and $(n - \varkappa + 1)^{st}$ terms of f' vanish,
by $T_{r,n-\varkappa}$ the first term of $C_{n-r,\varkappa}$ and the last term of $C_{r,n-\varkappa}$ vanish.

Using these results in the applications, we can determine the form of the desired covariant in each particular case.

§4.—APPLICATIONS OF THE THEORY.

1) *The Cubic.*

$n = 3$, $n - r > \varkappa$, hence the only values for r and $\varkappa$ are $r = \varkappa = 1$.

$$f_{(\mu)^2} = \alpha_z \alpha_\mu^2,$$
$$f_{(\mu)} = \alpha_z^2 \alpha_\mu.$$
$$\therefore \overline{R}_1 = \alpha_\mu \beta_\mu^2 \gamma_\mu^2 (\alpha\beta)(\alpha\gamma)$$
$$= \tfrac{1}{2} f. (\alpha\beta)^2 \alpha_\mu \beta_\mu.$$

or $\overline{C}_1$[*] $= (\alpha\beta)_2 \alpha\ \ \beta_\mu = \Delta$ of Clebsch.

If $\overline{C}_1 = (\alpha\beta)^2 \alpha_\mu \beta_\mu = (z'\xi)(z'\eta)$, we can give to λ and μ the values of the roots of $\overline{C}_1$.

Hence f can be transformed by $\overline{T}_1$ in two ways so that its 2nd and 3rd terms shall vanish.

2) *Quartic.*

there are two cases, $r = \varkappa = 1$; $r = 1$, $\varkappa = 2$.

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[*] Always omitting the numerical factor since we are to consider only the roots of the covariant.

a) $r = \varkappa = 1$.

$$\begin{aligned} f_{(\mu)^3} &= \alpha_z \alpha_\mu^3, \\ f_{(\mu)} &= \alpha_z^3 \alpha_\mu . \end{aligned}$$

$$\begin{aligned} \overline{R_1} &= \alpha_\mu \beta_\mu^3 \gamma_\mu^3 \delta_\mu^3 (\alpha\beta)(\alpha\gamma)(\alpha\delta), \\ &= f. \alpha_\mu \beta_\mu^2 \delta_\mu^3 (\alpha\beta)^2 (\alpha\delta), \\ &= f. T. \\ \overline{C_1} &= \alpha_\mu \beta_\mu^2 \delta_\mu^3 (\alpha\beta)^2 (\alpha\delta) = T \text{ of Clebsch.} \end{aligned}$$

By the six transformations $\overline{T_1}$ the 2nd and 4th terms of f vanish.

b) $r = 1$, $\varkappa = 2$.

$$\begin{aligned} f_{(\mu)^3} &= \alpha_z \alpha_\mu^3, \\ f_{(\mu)^2} &= \alpha_z^2 \alpha_\mu^2 . \\ R_{3,2} &= \alpha_\mu^2 \beta_\mu^3 \gamma_\mu^3 (\alpha\beta)(\alpha\gamma) \\ &= \tfrac{1}{2} f (\alpha\gamma)^2 \alpha_\mu^2 \gamma_\mu^2 = \tfrac{1}{2} f. H. \\ C_{3,2} &= (\alpha\gamma)^2 \alpha_\mu^2 \gamma_\mu^2 = H. \\ f_{(\lambda)} &= \alpha_z^3 \alpha_\lambda, \\ f_{(\lambda)^2} &= \alpha_z^2 \alpha_\lambda^2 . \end{aligned}$$

From §3, 4), *a*), if we apply a transformation of the set $T_{n-r,\kappa}$ to $C_{r,n-\kappa}$, its first coefficient will vanish. Also by the transformations $T_{n-r,\kappa}$ the $(n - r, + 1)^{\text{st}}$ and $(\varkappa + 1)^{\text{st}}$ terms of f' vanish.

But by actual trial $(Hi - fj)$ is the only combination of fundamental covariants whose first coefficient in the transformed form vanishes, and which is of the same degree and order as $C_{1,2}$.

Therefore $C_{1,2} = (Hi - fj)$, where i and j are the invariants of the quartic.

The quartic can be linearly transformed by the four transformations $T_{3,2}$ so that its 3rd and 4th terms vanish, and by $T_{1,2}$ so that its 2nd and 3rd terms vanish.

3) *Quintic.*

There are four cases; $r = \varkappa = 1$; $r = \varkappa = 2$; $r = 1$, $\varkappa = 2$; $r = 1$, $\varkappa = 3$.

a) $r = \varkappa = 1$.

$$\begin{aligned} f_{(\mu)^4} &= \alpha_z \alpha_\mu^4, \\ f_{(\mu)} &= \alpha_z^4 \alpha_\mu, \\ \overline{R_1} &= \alpha_\mu \beta_\mu^4 \gamma_\mu^4 \delta_\mu^4 \varepsilon_\mu^4 (\alpha\beta)(\alpha\gamma)(\alpha\delta)(\alpha\varepsilon). \end{aligned}$$

To express this in terms of fundamental covariants;* multiply

$$2\beta_\mu \gamma_\mu (\alpha\beta)(\alpha\gamma) = \beta_\mu^2 (\alpha\gamma)^2 + \gamma_\mu^2 (\alpha\beta)^2 - \alpha_\mu^2 (\beta\gamma)^2,$$

by $$2\delta_\mu \varepsilon_\mu (\alpha\delta)(\alpha\varepsilon) = \delta_\mu^2 (\alpha\varepsilon)^2 + \varepsilon_\mu^2 (\alpha\delta)^2 - \alpha_\mu^2 (\delta\varepsilon)^2,$$

and then by $$\alpha_\mu \beta_\mu^3 \gamma_\mu^3 \delta_\mu^3 \varepsilon_\mu^3 .$$

Assembling identical terms

$$\begin{aligned} 4\,\overline{R}_1 &= 4\alpha_\mu \beta_\mu^3 \gamma_\mu^5 \delta_\mu^3 \varepsilon_\mu^5 (\alpha\beta)^2 (\alpha\delta)^2 - 3f.\,4H^2, \\ &= 4f^3.\,S - 12f.\,H, \\ \overline{C}_1 &= f^2 S - 3H^2. \end{aligned} \qquad \text{where } \begin{aligned} \beta_\mu^3 \gamma_\mu^3 (\beta\gamma) &= 2H, \\ \alpha_\mu \beta_\mu (\alpha\beta)^4 &= 2S. \end{aligned}$$

$\overline{C}_1$ furnishes twelve transformations by which the 2nd and 5th terms of f' vanish.

b) $r = \varkappa = 2$.

$$\begin{aligned} f_{(\mu)^3} &= \alpha_z^2 \alpha_\mu^3, \\ f_{(\mu)^2} &= \alpha_z^3 \alpha_\mu^2, \end{aligned}$$

From §3, 4), *b*), by $\overline{T}_2$ the first and last coefficients of $\overline{C}_2'$ vanish; also by $\overline{T}_2$ the 3rd and 4th terms of f' vanish.

Therefore $\overline{C}_1 = (HI - fJ)$, where I is the covariant of the quintic of degree 2 and order 2, and J is the covariant of degree 3 and order 3.

c) $r = 1$, $\varkappa = 2$.

$$\begin{aligned} f_{(\mu)^4} &= \alpha_z \alpha_\mu^4, \\ f_{(\mu)^2} &= \alpha_z^3 \alpha_\mu^2 . \\ R_{4,2} &= \alpha_\mu^2 \beta_\mu^4 \gamma_\mu^4 \delta_\mu^4 (\alpha\beta)(\alpha\gamma)(\alpha\delta), \\ &= f\,\alpha_\mu^2 \beta_\mu^3 \delta_\mu^4 (\alpha\beta)^2 (\alpha\delta) \\ &= f.\,T. \\ C_{4,2} &= \alpha_\mu^2 \beta_\mu^3 \delta_\mu^4 (\alpha\beta)^2 (\alpha\delta) = T. \\ f_{(\lambda)} &= \alpha_z^4 \alpha_\lambda, \\ f_{(\lambda)^3} &= \alpha_z^2 \alpha_\lambda^3 . \end{aligned}$$

By the transformations $T_{4,2}$ the 3rd and 5th terms of f' vanish and the first coefficient of $C_{1,3}'$.

$$C_{1,3} = (fS^2 - 16HT) \text{ in Salmon's notation.}$$

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* Salmon, "Mod. Higher Alg.," p. 319.

d) $r = 1, \varkappa = 3$.

$$\begin{aligned} f_{(\mu)^4} &= \alpha_z \, \alpha_\mu^4 , \\ f_{(\mu)^3} &= \alpha_z^2 \, \alpha_\mu^3 ; \\ R_{4,3} &= \alpha_\mu^3 \, \beta_\mu^4 \, \gamma_\mu^4 \, (\alpha\beta) \, (\alpha\gamma). \\ &= \tfrac{1}{2} f. \, H. \\ C_{4,3} &= H. \\ f_{(\lambda)} &= \alpha_z^4 \, \alpha_\lambda , \\ f_{(\lambda)^4} &= \alpha_z^3 \, \alpha_\lambda^2 . \end{aligned}$$

By $T_{4,3}$ the 4th and 5th terms of f vanish and the first coefficient of $C'_{1,2}$.

Therefore $C_{1,2} = (I^3 - 27 \, J^2)$ where I and J are the same covariants mentioned before for the quintic.

4) *Sextic.*

There are six cases: $r = \varkappa = 1$; $r = \varkappa = 2$; $r = 1$, $\varkappa = 2$; $r = 1$, $\varkappa = 3$; $r = 2$, $\varkappa = 3$; $r = 1$, $\varkappa = 4$.

a) $r = \varkappa = 1$.

$$\begin{aligned} f_{(\mu)^5} &= \alpha_\mu \, \alpha_\mu^5 , \\ f_{(\mu)} &= \alpha_\mu^5 \, \alpha_\mu , \\ \overline{R}_1 &= \alpha_\mu \, \beta_\mu^5 \, \gamma_\mu^5 \, \delta_\mu^5 \, \varepsilon_\mu^5 \, i_\mu^5 \, (\alpha\beta) \, (\alpha\gamma) \, (\alpha\delta) \, (\alpha\varepsilon) \, (\alpha i). \end{aligned}$$

To express $\overline{R}_1$ in terms of fundamental covariants multiply*

$$(\alpha\beta) \, (\alpha\delta) \, \beta_\mu \, \delta_\mu = \tfrac{1}{2} \left[(\alpha\beta)^2 \delta_\mu^2 + (\alpha\delta)^2 \beta_\mu^2 - (\beta\delta)^2 \, \alpha_\mu^2 \right],$$

by

$$(\alpha\gamma) \, (\alpha\varepsilon) \, \gamma_\mu \, \varepsilon_\mu = \tfrac{1}{2} \left[(\alpha\gamma)^2 \varepsilon_\mu^2 + (\alpha\varepsilon)^2 \, \gamma_\mu^2 - (\gamma\varepsilon)^2 \, \alpha_\mu^2 \right],$$

and then by

$$\alpha_\mu \, \beta_\mu^4 \, \gamma_\mu^4 \, \delta_\mu^4 \, \varepsilon_\mu^4 \, i_\mu^5 \, (\alpha i) ;$$

we get

$$\begin{aligned} R_1 &= f \left[2 \, HT - f^2 S \right], \\ \overline{C_1} &= 2 \, HT - f^2 \, S. \end{aligned}$$

where

$$\begin{aligned} T &= - (\alpha\beta)^2 \, (\alpha i) \, \alpha_\mu^3 \, \beta_\mu^4 \, i_\mu^5 , \\ S &= - (\alpha\beta)^2 \, \alpha_\mu \, \beta_\mu^2 \, i_\mu^5 , \\ 2 \, H &= \quad (\gamma\varepsilon)^2 \, \gamma_\mu^4 \, \varepsilon_\mu^4 . \end{aligned}$$

b) $r = \varkappa = 2$.

$$\begin{aligned} f_{(\mu)^4} &= \alpha_\mu^2 \, \alpha_\mu^4 , \\ f_{(\mu)^3} &= \alpha_z^4 \, \alpha_\mu^4 . \end{aligned}$$

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* Stephanos, "Mém. des Sav. Etr.," Vol. XXVII, p. 116.

Even in this case the formula §3, 3) is complicated and it is not easy to express in fundamental covariants in such a way that f will divide out.

By $\overline{T}_2$ the 3rd and 5th terms of f' vanish and also the first and last coefficients of C'_2.

Therefore $$\overline{C}_2 = (fS^2 - 16\ HT).$$

c) $r = 1$, $\varkappa = 2$.

$$f_{(\mu)^5} = \alpha_z\,\alpha_\mu^5,$$
$$f_{(\mu)^2} = \alpha_z^4\,\alpha_\mu^2.$$
$$R_{5,2} = \alpha_\mu^2\,\beta_\mu^5\,\gamma_\mu^5\,\delta_\mu^5\,\varepsilon_\mu^5\,(\alpha\beta)\,(\alpha\gamma)\,(\alpha\delta)\,(\alpha\varepsilon).$$

By $T_{1,4}$ the 2nd and 5th terms of f' vanish and the 1st term of $C'_{5,2}$.

Therefore $C_{5,2} = (f^2S - 3\ H^2)$ Salmon's notation.

$$f_{(\lambda)} = \alpha_z^5\,\alpha_\lambda,$$
$$f_{(\lambda)^4} = \alpha_z^2\,\alpha_\lambda^4.$$

By $T_{5,2}$ the 3rd and 6th terms of f' vanish and the first term of $C'_{1,4}$.
Therefore

$$C_{1,4} = C_{1.6}^2 \,.\, C_{4.4} - 2\,C_{1.6}^2 \,.\, I_2 \,.\, C_{2.4} + 36\,C_{1.6} \,.\, C_{2.4}\,C_{3.6} + 3\,[C_{3.8}^2 + 4\,C_{2.4}^2 \,.\, C_{2.8}].$$

(Notation of E. B. Elliott in "Algebra of Quantics.")

d) $r = 1$, $\varkappa = 3$.

$$f_{(\mu)^5} = \alpha_z\,\alpha_\mu^5,$$
$$f_{(\mu)^3} = \alpha_z^3\,\alpha_\mu^3.$$
$$R_{5,3} = \alpha_\mu^3\,\beta_\mu^5\,\gamma_\mu^5\,\delta_\mu^5\,(\alpha\beta)\,(\alpha\gamma)\,(\alpha\delta),$$
$$= f.\ \alpha_\mu^3\,\beta_\mu^4\,\gamma_\mu^5\,(\alpha\beta)^2\,(\alpha\delta),$$
$$= f.\ T.$$
$$C_{5,3} = \alpha_\mu^3\,\beta_\mu^4\,\gamma_\mu^5\,(\alpha\beta)^2\,(\alpha\delta) = T.$$
$$f_{(\lambda)} = \alpha_z^5\,\alpha_\lambda,$$
$$f_{(\lambda)^3} = \alpha_z^3\,\alpha_\lambda^3.$$

$C_{1.3}$ is of degree twelve and order seven, but the covariants required to express it are of too high an order to handle easily.

e) $r = 2$, $\varkappa = 3$.

$$f_{(\mu)^4} = \alpha_z^2\,\alpha_\mu^4,$$
$$f_{(\mu)^3} = \alpha_z^3\,\alpha_\mu^3.$$

By $T_{2,3}$ the 3rd and 4th terms of f' vanish and the first term of $C'_{4,3}$. Therefore

$$C_{4.3} = HS - fT. \quad \text{(Salmon's notation).}$$

f). $r = 1$, $\varkappa = 4$.

$$\begin{aligned} f_{(\mu)^5} &= \alpha_z \alpha_\mu^5, \\ f_{(\mu)^4} &= \alpha_z^2 \alpha_\mu^4. \\ R_{5,4} &= f.\, H, \\ C_{5,4} &= H. \\ f_{(\lambda)} &= \alpha_z^5 \alpha_\mu, \\ f_{(\lambda)^2} &= \alpha_z^4 \alpha_\mu^2. \end{aligned}$$

In this case $C_{1,2}$ is complicated and requires covariants of a high order to express it.

Whenever the 2nd and 3rd terms of a quantic vanish $C_{n-r,\kappa}$ is always the Hessian, and whenever the 2nd and 4th terms of f vanish $C_{n-r,\kappa}$ is the Jacobian or T.

In general when two adjacent terms of a quantic vanish $C_{n-r,\kappa}$ and $C_{r,n-\kappa}$ will be Hessians of some order; and when only one term appears between the two which vanish, then $C_{n-r,\kappa}$ and $C_{r,n-\kappa}$ will be Jacobians of some order.

A Hessian of higher order may be defined up to a numerical factor as the result of eliminating z between.

$$\begin{matrix} \alpha_z^r \alpha_\mu^{n-r} \\ \alpha_z^{r+1} \alpha_\mu^{n-r-1} \end{matrix} \qquad r = (1, 2, \ldots . n - 2)$$

and then dividing out $f = \alpha_\mu^n$.

Similarly a Jacobian of higher order may be defined up to a numerical factor as the result of eliminating z between.

$$\begin{matrix} \alpha_z^r \alpha_\mu^{n-r} \\ \alpha_z^{r+2} \alpha_\mu^{n-r-2} \end{matrix} \qquad (r = 1, 2, \ldots . n - 3),$$

and then dividing out $f = \alpha_z^n$.

When $r = 1, 2, 3 \ldots .$ etc., we have Hessians and Jacobians of the 1st, 2nd, 3rd .... etc., orders respectively.

These examples comprise all the canonical forms, in which two terms vanish simultaneously by linear transformation, of the non-singular cubic, quartic, quintic and sextic.

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